The Magic Rhombicube

By Giuseppe Gori – Aug. 19, 2019
https://medium.com/@gori70/the-amazing-magic-rhombicube-7519074057ab

[PART 1] Last week, I discovered the amazing properties of a planar representation of a solid cube which I call a **rhombicube**. It is a flat, 2-dimensional figure made of rhombi oriented in three different directions to form a hexagonal shape (This is not to be confused with a hemicube, or half-cube, which is an abstract solid polyhedron with three square faces).

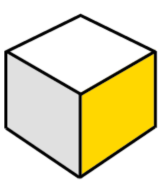
Patterns of rhombicubes were used since the time of the ancient Romans for tiling arrangements and they also occurs in basket making.

The rhombi's 3-directional orientation makes them appear as 3D cubes with three visible faces and three hidden faces.

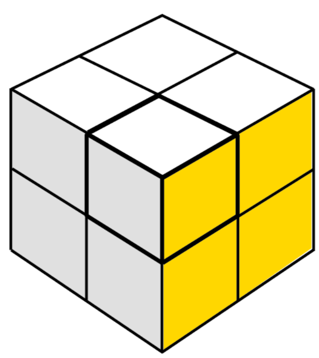
Mosaic floor, House of the Faun, Pompei

Basket making example

The simplest rhombicube is composed of only three rhombi as it appears on the figures above. The next simplest, the order-2 rhombicube, is shown below on the right.



The order-1 rhombicube



12 rhombi form the order-2 rhombicube

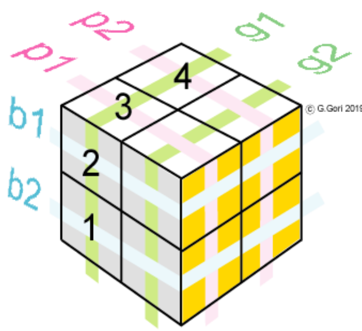
The order-1 rhombicube has no "magic" solution. We will focus instead on the order-2 rhombicube, composed by 12 equal rhombi, or cells, oriented in groups of four (its faces).

Notice that an order-1 rhombicube appears *at the center* of our order-2 rhombicube.

Our rhombicube will be filled with the numbers from 1 to 12, one number for each cell.

We divide our cells into six groups of four cells, marked below by a colored ribbon.

Because of their 3D appearance, we can also describe these groups of cells as the faces of a slice of a Rubik's cube. We can imagine two slices for each of the three apparent dimensions:



Identifying groups of 4 cells

In the figure above the numbers 1 to 4 are placed in the cells of slice g1. Here the color (green) identifies one of the three directions we could slice the imaginary cube in 3D.

Each one of the six colored ribbons b1, b2, p1, p2, g1 and g2 identifies a group of four cells.

Can our rhombicube become "magic"? That is, can we arrange its numbers so that all six groups of four cells have numbers that add up to the same magic constant?

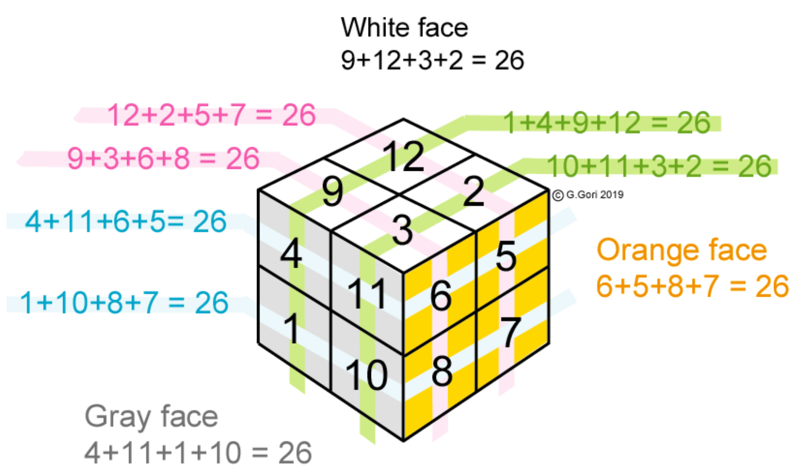
The answer is yes! Our magic constant is 26.

As we looked for solutions, not only *we found magic solutions*, but we discovered a ***remarkable feature*** of a Magic Rhombicube:

Surprise number 1:

> *When all six groups of four cells have numbers that add up to 26, then **necessarily** the sums of the four numbers on each face - white, gray and orange, **also add up to 26!***

For example:



Order-2 Magic Rhombicube: Necessarily its faces add up to 26

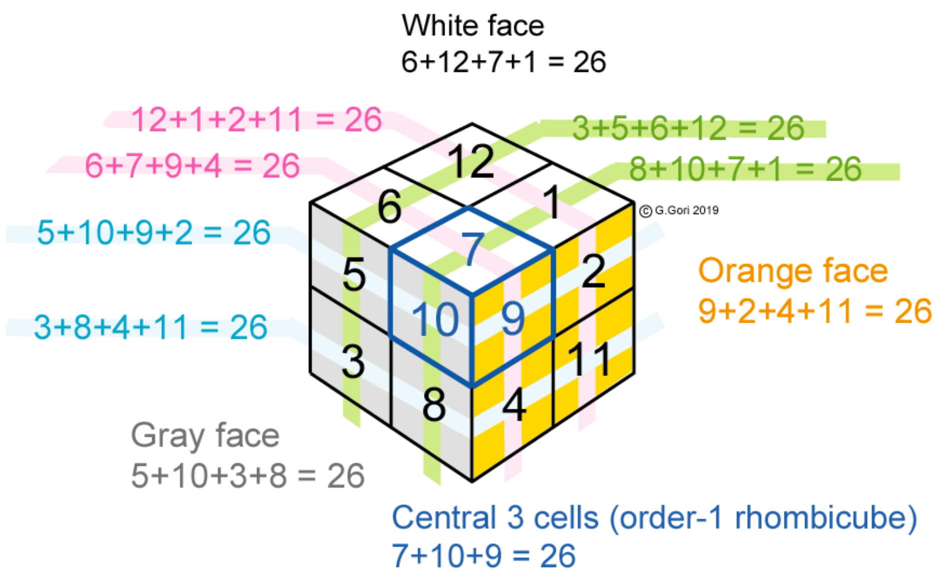
It looks like the face numbers want to rearrange themselves in an orderly fashion without us even asking! Just this discovery is worth sharing.

But wait, that's not all. Another surprise awaits us.

Surprise number 2:

> *We can rearrange the numbers on our rhombicube so that the numbers in the three cells at its center, themselves an order-1 rhombicube,* ***also add up to 26****.*

Now that's truly magic!



Central order-1 rhombicube: Its numbers now add up to 26

This implies that the numbers in black in the above figure, in the external cells at the perimeter of our hexagon, always add up to 52 - the sum of the numbers from 1 to 12 (78) minus 26.

Oh, did I miss something?

Did you notice that only three cells of our rhombicube are not adjacent to the central order-1 rhombicube?

These cells completely own three of the six corners of the external hexagon, which is the perimeter of the rhombicube. In the above figure, these cells have numbers 3, 11 and 12.

Surprise number 3: (Well, you may have guessed)

> *The numbers in these three corner-owning cells* ***also add up to 26!***

OK, now brace yourself.

What if I told you that we can find one more group of cells in a symmetrical arrangement, that have numbers adding up to our magic constant?

Look at the diagram above. Can you see it? It should not be difficult.

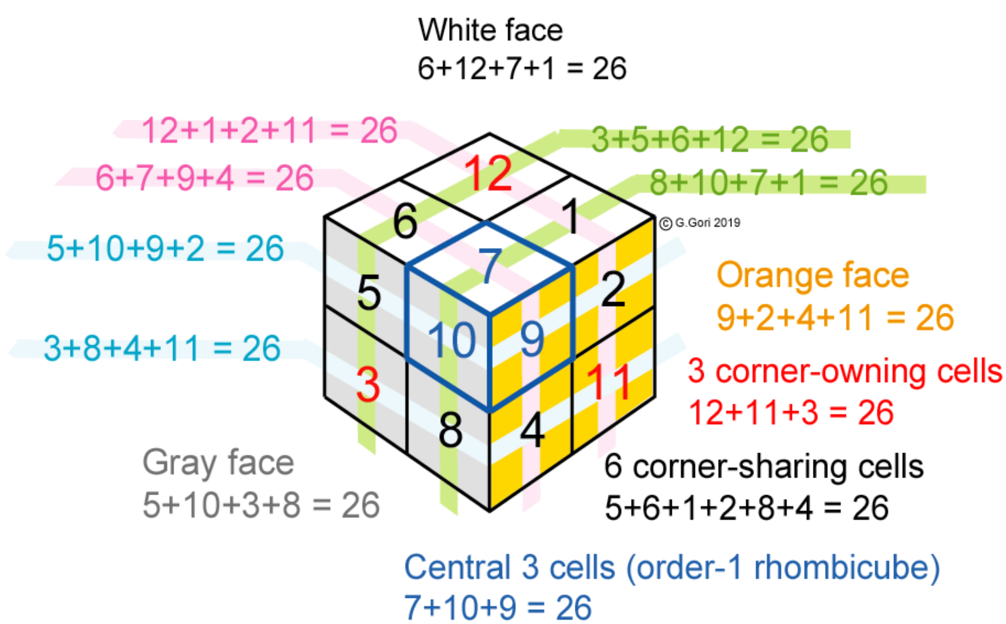
The remaining three corners of the external hexagon of our rhombicube are bisected by an edge line shared by two cells. We call these corner-sharing cells.

That's six corner-sharing cells in total.

Surprise number 4:

The numbers in these corner-sharing cells (5+6+1+2+8+4) ***also add up to 26****.*

I ran out of exclamation marks.



The amazing Magic Rhombicube

Our Magic Rhombicube is one of the simplest magic figures. It has just 3 more cells than the classic 3x3 Magic Square.

It is a "normal" magic figure, as it uses the consecutive numbers 1 to 12.

By only using twelve numbers we obtain its magic constant, 26, in ***twelve different ways.***

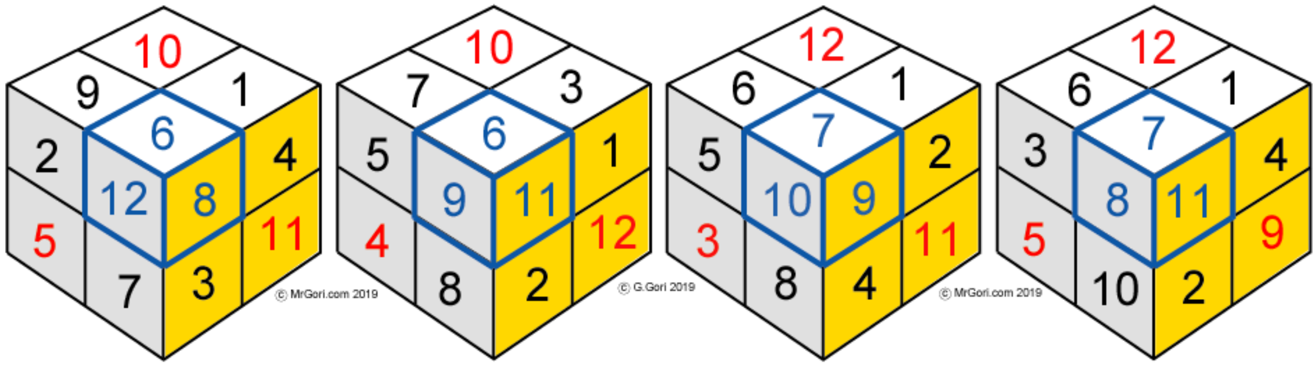
It seems that each number has its own magic place.

Our Magic Rhombicube helps us visualize the different ways in which we can group numbers from a series and how these groups intersect each other.

The Magic Rhombicube can be used in education to enhance the students' ability to think, observe and recognize patterns. For more on this, please keep reading...

[PART 2] The Magic Rhombicube solutions and properties

However surprising the above solution may be, it is one of 72 possible solutions. We isolated four solutions that are not obvious transformations of each other:



Magic Rhombicube: Set of representative solutions

These four particular solutions are representative of groups of analogous solutions.

All together, the solutions to our Magic Rhombicube provide us with only a subset of all the ways in which four numbers, out of the series 1 to 12, can add up to 26 without repetition. Some combinations appear many times, while others (e.g., 1+5+8+12=26 or 3+4+7+12=26) never appear.

Nevertheless, our 72 solution variations include unexpected patterns and properties.

Slice and face group properties

In each of our solutions, there are nine groups of four cells with numbers adding up to 26.

We initially identified **six** groups of four cells with our colored ribbons. We called these slices. We then saw that there were **three** more groups of four cells, the rhombicube faces, also with numbers adding up to 26.

However, these groups have different properties, as they intersect with each other differently:

- Face groups intersect only four slice groups and no other face group.
- Slice groups do not intersect with the slice group in the same direction, but intersect the four slice groups in the other directions plus they intersect with two face groups.

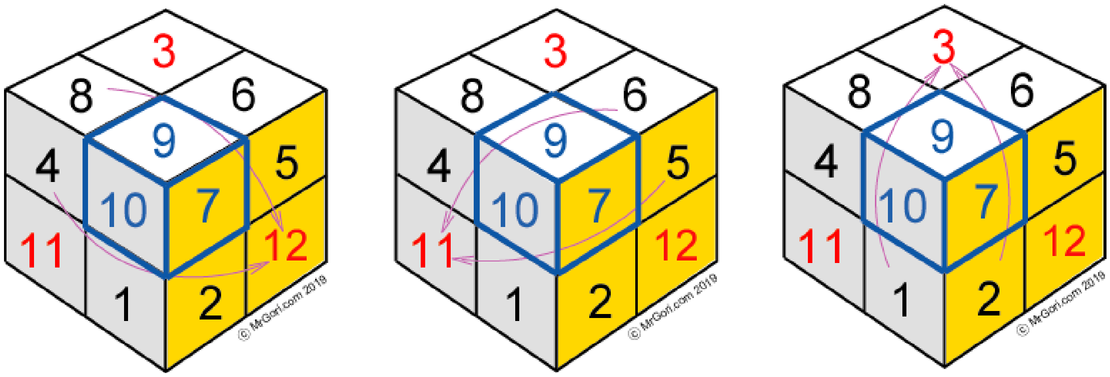
The following properties are observed **in every solution** of our Magic Rhombicube:

The first solution we presented, which uncovered its first surprise, although itself remarkable, did not show us the Magic Rhombicube's full potential.
After we rearranged the numbers and required the three central cells to add up to our magic sum, we discovered more groups adding to our magic sum.
Without raising our requirements our journey of discovery would have been cut short.
Now instead, we are about to uncover even more properties which would not have occurred had we stopped at our first solution.

Property 1:

The number in a cell-owning corner of the external hexagon forming the perimeter of our rhombicube is always the sum of the two numbers in the opposite, corner-sharing cells.

This property is quite easy to verify, but not easy to notice!
For example, in the solution below, 4+8=12, 6+5=11 and 1+2=3.



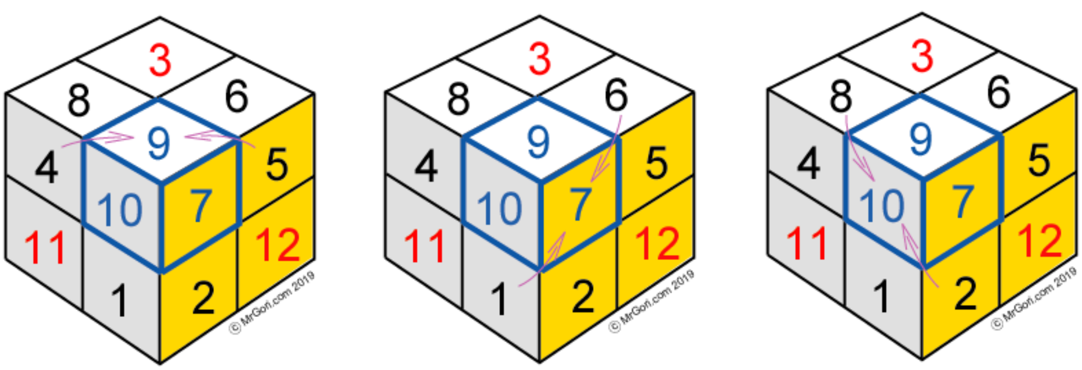
Property 1: Opposite corners' sums

The three numbers involved (the two addends and their sum), in each of the three corner cells, are each in a different face of the rhombicube. They are also part of three slices in each of the three directions. That's a lot of constraints being satisfied!

As we have seen, there are three cells forming the central order-1 rhombicube. A similar sum property is observed with each of the numbers in these central cells:

Property 2:

If we pick one of the central cells, then its number is the sum of two addends. These can be found in the cells opposite to the acute angles of the central cell we picked.



Property 2: Central sums

As with Property 1, this property involves numbers in three different faces and in three different slices. All numbers are so tightly connected!

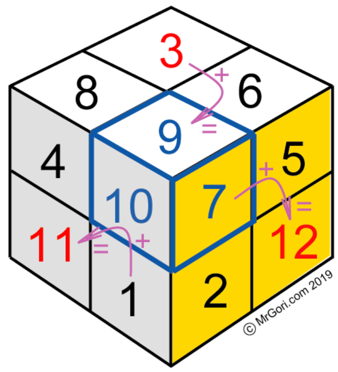
Property 3:

Face groups always contain a number that is the sum of two out of the other three.

For example, looking at the last diagram above, the first solution's gray face contains the numbers 2, 5, 7 and 12. Here we can see that 2+5=7.

If we pick the last orange face on the right, we have the numbers 2, 4, 9 and 11. Here 2+9=11.

We can verify this for all faces of every solution, but this property does not always apply to slice groups.



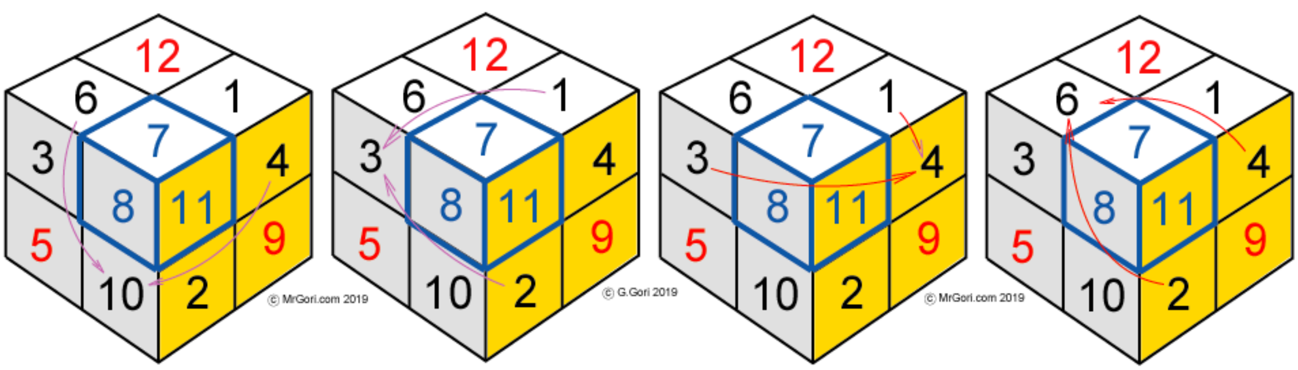
Property 3 of face groups

The following property has to do with the group of six corner-sharing cells (black numbers).

Property 4a:

Either one or two numbers of the corner-sharing cells group are the sum of two other numbers within the same group, such that the three numbers, addends and sum, are in three different faces and symmetrically arranged with respect to the center of the rhombicube.

In the following solution example, only the first two sums, 10 and 3, comply with this property.



The first two sums, 10 and 3, comply with Property 4a

When one complying sum total is found, this number can only be 3 or 6. When a complying sum total of 8, 9 or 10 is found, then a second complying sum occurs. The solutions with two complying sums are: **8 and 5**, or **9 and 4**, or **10 and 3** (as in the above example).

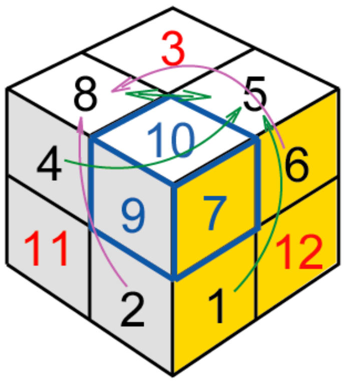
No other sum number or combination can be found that complies with this property.

The following is is a further observation about the group of six corner-sharing cells:

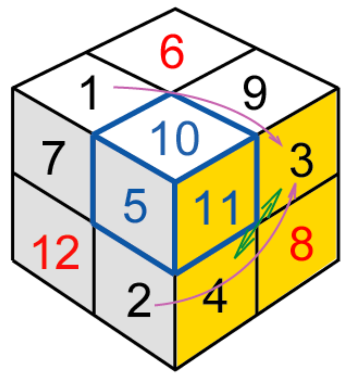
Property 4b:

When two sums complying to Property 4a are found, then the two sums are always in the same face and in opposite cells.
When only one sum is found complying with Property 4a, then

- *if the sum is 3, its opposite cell in the same face is 4.*
- *if the sum is 6, its opposite cell in the same face is 8.*



Two complying sums, 8 opposite to 5



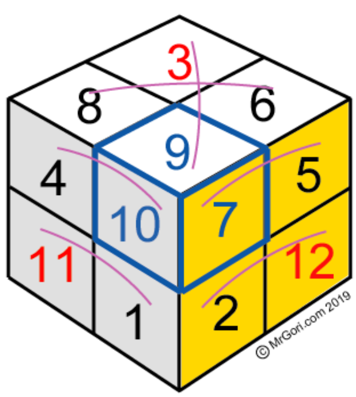
One complying sum, 3 opposite to 4

Property 4b

Property 5a:

In each solution, for each face, we can add its numbers in pairs in a way that the two results (demi-sums) ***are the same for all three faces****. These demi-sums can only be* ***7 and 19, 12 and 14 or twice 13****. In one face the numbers to be added are opposite to each other and in the other two faces the numbers are adjacent to each other.*

Obviously by adding these demi-sums 7+19, 12+14 and 13+13 we obtain the total for the face: 26. The following is an example of a solution with face demi-sums 12 and 14:



Face groups: Property 5a: Face demi-sums 12 and 14:

The demi-sums 3&23, 4&22, 5&21, 6&20, 8&18, 9&17, 10&16, 11&15 never appear in our solutions' faces, while all possible demi-sums, from 3&23 to 13&13 appear in slice groups.

You can see further examples of this property by looking at the set of four representative solutions presented earlier.

- in the first solution we find: white face (adjacent): 6+1=**7** and 10+9=**19** - gray face: (adjacent) 5+2=**7** and 12+7=**19** – orange face (opposite): 4+3=**7** and 11+8=**19.**
- in the other three solutions we find **both** demi-sums of **13** for each face: in two faces by adding adjacent numbers and, in the gray faces, by adding opposite cell numbers.

The following property has to do with the orientation of the demi-sums of property 5a:

Property 5b:

*The **adjacent** demi-sums observed in two faces according to Property 5a always occur in both faces in the same direction. Furthermore, this direction (in apparent 3D) is always the direction along the slice line parallel to the third face.*

Property 6:

Face groups always contain two even numbers and two odd numbers.

The other combinations that could be valid to obtain 26 (four even numbers and four odd numbers) never occur in face groups, although they do occur in slice groups.

Property 7 (implied by 5a and 6):

Every solution has one face with odd numbers and even numbers opposite to each other. The other two faces have adjacent even and adjacent odd numbers.

The following property has to do with the two groups of three cells (the *central rhombicube* cells and the *corner-owning* cells) and the group of six corner-sharing cells.

Property 8:

The numbers 1 and 2 never occur in three-cell groups.

The numbers 11 and 12 always appear in three-cell groups.

Conversely,

The numbers 11 and 12 never occur in the group of six corner-sharing cells.

The numbers 1 and 2 always appear in the group of six corner-sharing cells.

Unfriendly number pairs

*Most numbers do not have a problem with each other, but some number pairs cannot see each other. They can be on the same slice, but **are never on the same face** of our Magic Rhombicube.*

These are: ***1 and 3***, ***1 and 5***, **2 and 3**, ***5 and 6***, **7 and 8**, **8 and 12**, and **10 and 12**.

The role of the number 3 in the Magic Rhombicube

Our Magic Rhombicube has a 3-dimensional imprint. Because of this, it can show us the properties of groups better than a 2-dimensional Magic Square.

You may have noticed how the number 3 plays an important role in our Magic Rhombicube.

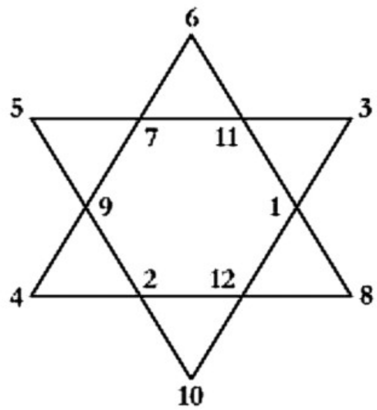
1. The rhombicube appearance is **3**-dimensional.
2. There are **3** groups of rhombi (tiles) oriented in **3** different ways, the only 3 ways possible to form a hexagon.
3. The total sum of the series of numbers 1 to 12 is 78. This divided by **3** gives us our magic constant.
4. The six groups identified by ribbons, or slices, are in **3** directions in space.
5. **3** of the six corners of the whole hexagon are corner-owning cells, The other **3** are corner-sharing cells.
6. The whole hexagon, containing all cells, can be divided into **3** faces of equivalent weight – each with numbers adding up to 26.
7. The same hexagon, containing all cells, can be divided into **3** zones (central order-1 rhombicube, corner-owning cells and corner-sharing cells) of equivalent weight - each with numbers adding up to 26.
8. Two of our groups (central rhombicube and corner-owning cells) are composed of **3** cells.
9. Each face contains **3** numbers where one is the sum of the other two.
10. The group of corner-sharing cells contains at least **3** numbers where one is the sum of the other two and complies with Property 4. These numbers are in **3** different faces.
11. Property 1 and Property 2 involve numbers that are in **3** corner cells or **3** central cells. The **3** numbers involved (the addends and their sum) are in all **3** different faces of the rhombicube. They are also part of **3** slices in each of the **3** directions.

[PART 3] A relative of the simplest Magic Rhombicube: The simplest Magic Star

In classic Magic Stars, as described in the literature, numbers are associated with *the intersections between two lines* and are not not written inside cells.

The simplest possible **normal** Magic Star is a hexagon-star of order-6, using the star of David basic shape. This was solved by H. E. Dudeney in 1926. It has 80 solutions.

Like our Magic Rhombicube, it also uses the numbers 1 to 12. Its magic constant is also 26:



The simplest Magic-Star

Relation between the Rhombicube and the simplest Magic Star

Each segment in this Magic Star identifies a groups of four numbers. Since these groups add up to 26, we can see the relations between the simplest Magic Star and the simplest Magic Rhombicube.

The simplest Magic Star identifies six such groups, one for each of its segments. This grouping is analogous to the grouping defined by the six slices in our rhombicube.

However, each rhombicube solution is more fertile as it shows us, with its faces, three more four-number group combinations adding up to the magic constant.

Furthermore, each rhombicube solution presents us with the surprises of two groups of three numbers and a group of six numbers also adding up to the same magic constant.

Like all other groups in our rhombicube, these extra groups are visually highlighted by their symmetric arrangement.

Finally, our Magic Rhombicube can show us the amazing properties of its numbers arranged in many symmetrical and interesting patterns.

Other Known Magical Figures

There are other, more complex magic figures (see links below). However, uniqueness and beauty are found in the simplest patterns displaying the most unexpected properties.

Interesting Links

For more on **Magic Stars** see:

http://magic-squares.net/magic_stars_index.htm
http://recmath.org/Magic%20Squares/order6.htm
https://math.info/Misc/Magic_Star/

For **Magic Squares** see:

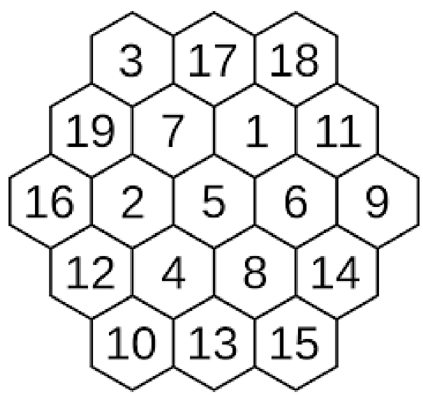
https://en.wikipedia.org/wiki/Magic_square
http://magic-squares.net/magic_squares_index.htm

6	**1**	**8**
7	**5**	**3**
2	**9**	**4**

The smallest, normal **Magic Hexagon** is the unique 19-cell figure below.

Here the five rows, in each direction, have a different number of cells.

See: http://mathworld.wolfram.com/MagicHexagon.html



The only normal Magic Hexagon

Numbers: 1 to 19. Magic constant: 38

Giuseppe Gori has a doctorate degree in Computer Science from the University of Pisa, Italy. He has been Assistant Professor of Computer Science at the University of Pisa and visiting professor of computer communication and networking at Western University, Ontario, Canada. He has worked for several large companies in the IT industry. He is the CEO of Gorbyte (https://gorbyte.com), an Ontario, Canada company pioneering in blockchain research, development and innovation.